VAR and SVAR II: Some equations

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Abstract

This script reviews some basic equations concerning VAR impulse-response functions. VAR(2) $\mathbf{y}_t = \Theta_1 \mathbf{y}_{t-1} + \Theta_2 \mathbf{y}_{t-2} + \mathbf{e}_t$ used as illustration.

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1 Impulse-response I (VMA)

▷ IRF for $\mathbf{y}_t = \Theta_1 \mathbf{y}_{t-1} + \Theta_2 \mathbf{y}_{t-2} + \mathbf{e}_t$ via VMA(∞)

$$\begin{aligned}
\mathbf{y}_t &= \Theta_1 \mathbf{y}_{t-1} + \Theta_2 \mathbf{y}_{t-2} + \mathbf{e}_t \\
\mathbf{y}_t &= \left(\mathbf{I} - \Theta_1 L - \Theta_2 L^2\right)^{-1} \mathbf{e}_t \\
\mathbf{y}_t &= \left(\mathbf{I} + \Psi_1 L + \Psi_2 L^2 + \ldots\right) \mathbf{e}_t
\end{aligned}$$

▷ Calculation of Ψ elements

$$\begin{aligned}
\left(\mathbf{I} - \Theta_1 L - \Theta_2 L^2\right)^{-1} &= \left(\mathbf{I} + \Psi_1 L + \Psi_2 L^2 + \ldots\right) \\
\mathbf{I} &= \left(\mathbf{I} - \Theta_1 L - \Theta_2 L^2\right)\left(\mathbf{I} + \Psi_1 L + \Psi_2 L^2 + \ldots\right) \\
\mathbf{I} &= \mathbf{I} + \left(\Psi_1 - \Theta_1\right) L \\
&\quad + \left(\Psi_2 - \Theta_1 \Psi_1 - \Theta_2\right) L^2 \\
&\quad + \left(\Psi_3 - \Theta_1 \Psi_2 - \Theta_2 \Psi_1\right) L^3 \\
&\quad + \left(\Psi_4 - \Theta_1 \Psi_3 - \Theta_2 \Psi_2\right) L^4 + \ldots
\end{aligned}$$

▷ Each parenthesis has to equal zero

$$\begin{aligned}\Psi_1 &= \Theta_1\\ \Psi_2 &= \Theta_2+\Theta_1^2\\ \Psi_3 &= 2\Theta_1\Theta_2+\Theta_1^3\\ \Psi_4 &= \Theta_2^2+3\Theta_1^2\Theta_2+\Theta_1^4\\ &\vdots\end{aligned}$$

2 Impulse-response II (VAR)

▷ Impulse-response functions (IRF) can also be represented by the autoregressive structure of the original model, e.g. $IRF_t = \Theta_1 IRF_{t-1}+\Theta_2 IRF_{t-2}$ using the autoregressive structure, $\Psi_1=\Theta_1$, and Ψ_0= $\mathbf{I}$. Calculation produces the same IRF

$$\begin{aligned}\Psi_2 &= \Theta_1\Psi_1+\Theta_2\Psi_0\\ &= \Theta_1^2+\Theta_2\end{aligned}$$

$$\begin{aligned}\Psi_3 &= \Theta_1\Psi_2+\Theta_2\Psi_1\\ &= 2\Theta_1\Theta_2+\Theta_1^3\end{aligned}$$

$$\begin{aligned}\Psi_4 &= \Theta_1\Psi_3+\Theta_2\Psi_2\\ &= \Theta_2^2+3\Theta_1^2\Theta_2+\Theta_1^4\\ &\vdots\end{aligned}$$

▷ Note that each equation $\Psi_t=\Theta_1\Psi_{t-1}+\Theta_2\Psi_{t-2}$ corresponds to each L parenthesis in the previous calculation

▷ Interpretation. AR elements say how the contemporaneous value is related to past elements. After a shock, that is normal to see the reaction of the vector being linked to the past reactions in the same way as without any shock

3 Impulse-response III (Companion)

▷ Companion way to calculate impulse-response functions

$$\begin{aligned}\mathbf{y}_t &= \Theta_1\mathbf{y}_{t-1}+\Theta_2\mathbf{y}_{t-2}+\mathbf{e}_t\\ \begin{pmatrix}\mathbf{y}_t\\ \mathbf{y}_{t-1}\end{pmatrix} &= \begin{pmatrix}\Theta_1 & \Theta_2\\ \mathbf{I} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{y}_{t-1}\\ \mathbf{y}_{t-2}\end{pmatrix}+\begin{pmatrix}\mathbf{e}_t\\ \mathbf{0}\end{pmatrix}\\ \boldsymbol{\varsigma}_t &= \mathbf{M}\boldsymbol{\varsigma}_{t-1}+\boldsymbol{\epsilon}_t\\ \text{where } \boldsymbol{\varsigma}_t &= \begin{pmatrix}\mathbf{y}_t\\ \mathbf{y}_{t-1}\end{pmatrix}, \mathbf{M}=\begin{pmatrix}\Theta_1 & \Theta_2\\ \mathbf{I} & \mathbf{0}\end{pmatrix}, \boldsymbol{\epsilon}_t=\begin{pmatrix}\mathbf{e}_t\\ \mathbf{0}\end{pmatrix}\end{aligned}$$

▷ Idea. This companion form corresponds to a VAR(1) with IRF using the single AR element $\mathbf{M}$ raised at a power corresponding to the impulse horizon. Only the 1,1 element is however relevant. Accordingly we use matrix $\mathbf{J} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \end{pmatrix}$ to select this element

▷ IRF

$$\begin{aligned}
IRF_1 &: \mathbf{JMJ}' = \Theta_1 \\
IRF_2 &: \mathbf{JM}^2\mathbf{J}' = \begin{pmatrix} \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \Theta_1^2 + \Theta_2 & \Theta_1\Theta_2 \\ \Theta_1 & \Theta_2 \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix} \\
&= \Theta_1^2 + \Theta_2 \\
IRF_3 &: \mathbf{JM}^3\mathbf{J}' = \begin{pmatrix} \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \Theta_1^3 + 2\Theta_1\Theta_2 & \Theta_1^2\Theta_2 + \Theta_2^2 \\ \Theta_1^2 + \Theta_2 & \Theta_1\Theta_2 \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix} \\
&= \Theta_1^3 + 2\Theta_1\Theta_2 \\
IRF_4 &: \mathbf{JM}^4\mathbf{J}' = \begin{pmatrix} \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \Theta_1^4 + 3\Theta_1^2\Theta_2 + \Theta_2^2 & \cdots \\ \vdots & \ddots \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix} \\
&= \Theta_1^4 + 3\Theta_1^2\Theta_2 + \Theta_2^2 \\
&\vdots
\end{aligned}$$

References

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