

Footnote in script, VAR(2)

Form of the matrix $\mathbf{A}(L)$ (equation 1.8_{walsh}) in case of a VAR(2); SVAR(2) has the following expression

$$\mathbf{A}_0 \mathbf{y}_t^* = \mathbf{A}_1 \mathbf{y}_{t-1}^* + \mathbf{A}_2 \mathbf{y}_{t-2}^* + \mathbf{e}_t$$

reduced form, pre-multiplying by \mathbf{A}_0^{-1}

$$\mathbf{y}_t^* = \mathbf{A}_0^{-1} \mathbf{A}_1 \mathbf{y}_{t-1}^* + \mathbf{A}_0^{-1} \mathbf{A}_2 \mathbf{y}_{t-2}^* + \underbrace{\mathbf{A}_0^{-1} \mathbf{e}_t}_{\mathbf{u}_t}$$

rewrite using lag operator for lags older than $t - 1$

$$\begin{aligned} \mathbf{y}_t^* &= \mathbf{A}_0^{-1} \mathbf{A}_1 \mathbf{y}_{t-1}^* + \mathbf{A}_0^{-1} \mathbf{A}_2 (\mathbf{L} \mathbf{y}_{t-1}^*) + \mathbf{u}_t \\ \mathbf{y}_t^* &= \mathbf{A}_0^{-1} (\mathbf{A}_1 + \mathbf{A}_2 \mathbf{L}) \mathbf{y}_{t-1}^* + \mathbf{u}_t \end{aligned}$$

which gives with $\mathbf{A}(L) = \mathbf{A}_0^{-1} (\mathbf{A}_1 + \mathbf{A}_2 \mathbf{L})$

$$\mathbf{y}_t^* = \mathbf{A}(L) \mathbf{y}_{t-1}^* + \mathbf{u}_t \quad (1.8_{\text{Walsh}})$$

Cholesky decomposition

Elements

Decomposition $\mathbf{u}_t = \mathbf{B}\mathbf{e}_t$, after calculation of the variance-covariance matrix $\text{Var}\mathbf{u}_t = \text{Var}(\mathbf{B}\mathbf{e}_t)$

$$\begin{bmatrix} \otimes & \otimes & \otimes & \otimes & \dots & \otimes \\ \otimes & \otimes & \otimes & \otimes & & \vdots \\ \otimes & \otimes & \otimes & \otimes & & \vdots \\ \otimes & \otimes & \otimes & \otimes & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \otimes & \dots & & \dots & \dots & \otimes \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ \otimes & 1 & 0 & 0 & & \vdots \\ \otimes & \otimes & 1 & 0 & & \vdots \\ \otimes & \otimes & \otimes & 1 & & \vdots \\ \vdots & & & & \ddots & 0 \\ \otimes & \dots & & \dots & \otimes & 1 \end{bmatrix} \begin{bmatrix} \otimes & 0 & 0 & 0 & \dots & 0 \\ 0 & \otimes & 0 & 0 & & \vdots \\ 0 & 0 & \otimes & 0 & & \vdots \\ 0 & 0 & 0 & \otimes & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & & \dots & 0 & \otimes & \end{bmatrix} \mathbf{B}'$$

Restrictions

Number of restrictions for a SVAR with p lags and n variables; the VAR estimation gives p matrices with n^2 elements and a variance-covariance matrix of the residuals with $\frac{n^2+n}{2}$ different elements (together $pn^2 + \frac{n^2+n}{2}$); for the SVAR model, we need to estimate $(p+1)n^2 + \frac{n^2+n}{2}$ elements; the number of restrictions is the difference $(p+1)n^2 + \frac{n^2+n}{2} - pn^2 - \frac{n^2+n}{2} = n^2$

Prisoner's dilemma, game theory, chapter 8

Setup		Zentralbank	
		Preisstabilität	Andere Ziele
Wirtschaftssubjekte	tiefe π^e	10, 10	4, 11
	hohe π^e	9, 4	5, 5

Nash-GG		Zentralbank	
		Preisstabilität	Andere Ziele
Wirtschaftssubjekte	tiefe π^e	<u>10</u> , 10	4, <u>11</u>
	hohe π^e	9, 4	<u>5</u> , <u>5</u>

Overview, complements, chapter 8

- Starting point: (actually end point, because the time inconsistency literature has contributed a lot to the following facts)
 - Microfunded models: New Keynesian model assumes a feedback rule à la Taylor with inflation and output gaps (stabilization of variables)
 - Reality: central banks have multiple objectives, with/without hierarchy; central banks behave with more or less discretion/commitment
- Problem: central banks may face incentives which lead to a suboptimal outcome (e.g. inflation and a certain GDP level, which could be achieved without inflation)
- Basic framework: game theory, static, simultaneous games
- Timing:

– Timing of the game: $\left\{ \begin{array}{cccccc} \text{i)} & & \text{ii)} & & \text{iii)} & & \text{iv)} & & \text{v)} \\ \pi^e & \rightarrow & e & \rightarrow & \Delta m & \rightarrow & v & \rightarrow & y, \pi \\ & & & & & & & & \text{simul.} \end{array} \right.$

- central bank observes π^e
- central bank observes e
- Timing (discretion):
 - * 1. Central bank finds Δm_t as function of π^e and e
 - * 2. Agents form π^e knowing how the central bank behaves, without seeing e
 - * 3. Central banks set Δm_t
 - * 4. Outcome y, π
- Timing (commitment):
 - * 1. Form of the rule is known, some parameters unknown

- * 2. Agents form π^e knowing the form of the rule
- * 3. Central bank optimally sets the parameters of the rule (minimizes its loss function over the rule parameters) and so Δm_t
- * 4. Outcome y, π

□ Utility function:

$$- U_t = \underbrace{\lambda(y_t - y_n)}_{\text{output expansion}} - \underbrace{\frac{1}{2}\pi_t^2}_{\text{inflation stabilization}}$$

- Form of the preference (U_t) influences the behavior of the bank, i.e. its discretionary policy or its optimal commitment
- Discretion:
 - * MB > MC, in equilibrium $\pi_t = a\lambda$
- Commitment: $\Delta m_t = 0$ (optimal parameters already given)
- $\mathbb{E}U_t^c > \mathbb{E}U_t^d$

□ Loss function:

$$- V_t = \underbrace{\frac{1}{2}\lambda(y_t - y_n - k)^2}_{\text{output stabilization}} + \underbrace{\frac{1}{2}\pi_t^2}_{\text{inflation stabilization}}$$

- Form of the preference (U_t) influences the behavior of the bank, i.e. its discretionary policy or its optimal commitment
- Discretion:

$$* \text{ Relation between inflation and expected inflation: } \pi_t = \underbrace{\frac{a^2\lambda}{1+a^2\lambda}}_{\text{slope} < 1} \pi_t^e + \underbrace{\frac{a\lambda k}{1+a^2\lambda}}_{\text{intercept} > 0},$$

in equilibrium $\pi_t = a\lambda k$

- Commitment: $\Delta m_t^c = 0 - \frac{a\lambda}{1+a^2\lambda} e_t$ (optimal commitment determined by the central bank)
- $\mathbb{E}V_t^c < \mathbb{E}V_t^d$ [not always true e.g. if forced commitment; volatility of output may play a role]

□ Solution against bias

- Repeated game (reputation of the central bank)
 - * Trigger strategy
 - * Gain of ‘cheating’ vs. cost of ‘cheating’
- Conservative banker (reputation of the central banker)
 - * More weight on inflation punishment in the loss function ($1 + \delta$)
 - * Same framework as weight on inflation 1 (loss function V_t)
 - * Optimal δ
- Contract with compensation
 - * Wage structure of the central banker, so he/she avoids the inflation bias, but keeps the optimal reaction to output volatility
- Flexible targeting rules -> relation to conservative banker and contract
- Strict targeting rules (monetary targeting, income targeting)

Overview Models

- At the SNB's (p. 23/37)
 - Non-structural
 - * i) Monetary indicators, financial market information, etc.
 - * ii) VAR and VEC, vector autoregressive model, vector error correction model
 - Semi-structural
 - * i) SVAR
 - * ii) SVEC
 - Structural
 - * i) SEM, simultaneous equation models, poorly microfunded
 - * ii) DSGE, dynamic stochastic general equilibrium model, New Keynesian, microfunded

- Structural
 - SEM, big models, Cowles Commission, 1930s (poor microfundation, poor shock behavior, essentially backward looking)
 - [RBC, real business cycles model, no money, all flexible, no rigidities, technical (productivity) shock, forward looking
 - [Sidrauski (MIU, \neq CIA), money, no rigidities, LR properties]
 - New Keynesian, DSGE models, a lot of rigidities, a lot of shocks, SR effects of monetary policy, no LR effect

Steps

- Script part 8.4.2.2: three-equation NK model
 - i) AD (IS)
 - ii) AS (PC)
 - iii) central bank behavior

TABLE 23 NK MODEL OVERVIEW

Model equations	
$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma} (\hat{i}_t - \mathbb{E}_t \pi_{t+1}) + u_t$	$\rightarrow -\frac{1}{\sigma} \hat{i}_t + \mathbb{E}_t x_{t+1} + \frac{1}{\sigma} \mathbb{E}_t \pi_{t+1} = x_t - u_t$
$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t$	$\rightarrow \beta \mathbb{E}_t \pi_{t+1} = -\kappa x_t + \pi_t$
$\hat{i}_t = \delta \pi_t + v_t$	$\rightarrow \hat{i}_t = \delta \pi_t + v_t$

- Microfunded models
 - Variables (states vs. controls), parameters
 - Rational, optimal behavior of agents (max/min under constraint); timing $t, t - 1, t + 1$
 - Set of equations (definition and behavioral equations)
 - Linearization of equations

 - (Recursive) solution of the model (t as a function of $t - 1$)
 - Calibration vs. estimation of the parameters (script p. 273)

Solution Example Question 8

Equilibrium inflation rate

Utility of conservative central banker, with supply curve y

$$\begin{aligned} U &= -(y - k)^2 - (1 + \delta) \pi^2 \\ U &= -(\pi - \pi^e + \varepsilon - k)^2 - (1 + \delta) \pi^2 \end{aligned}$$

group elements in the first parenthesis

$$U = -((\pi - \pi^e) + (\varepsilon - k))^2 - (1 + \delta) \pi^2$$

use formula $(a + b)^2 = a^2 + b^2 + 2ab$

$$U = -((\pi - \pi^e)^2 + (\varepsilon - k)^2 + 2(\varepsilon - k)(\pi - \pi^e)) - (1 + \delta) \pi^2$$

use formula again for the other square elements

$$\begin{aligned} U &= -\pi^2 - (\pi^e)^2 + 2\pi\pi^e - \varepsilon^2 - k^2 + 2k\varepsilon \\ &\quad - 2\varepsilon\pi + 2\varepsilon\pi^e + 2k\pi - 2k\pi^e - (1 + \delta) \pi^2 \end{aligned}$$

take expectation, knowing that the central banker observes ε ; $\mathbb{E}\varepsilon^2$ is the variance of the output shock

$$\begin{aligned} \mathbb{E}U &= -\pi^2 - (\pi^e)^2 + 2\pi\pi^e - \sigma_\varepsilon^2 - k^2 + 2k\varepsilon \\ &\quad - 2\varepsilon\pi + 2\varepsilon\pi^e + 2k\pi - 2k\pi^e - (1 + \delta) \pi^2 \end{aligned}$$

maximize the utility, take first derivative wrt π , and set it equal to zero

$$0 = -2\pi + 2\pi^e - 2\varepsilon + 2k - 2(1 + \delta) \pi$$

solve for the inflation rate as a function of the expected inflation rate by the economic agents

$$\pi = \frac{\pi^e - \varepsilon + k}{2 + \delta}$$

inflation expectations of agents, which do not observe ε

$$\mathbb{E}\pi = \frac{\pi^e - \mathbb{E}\varepsilon + k}{2 + \delta} = \pi^e$$

solve for π^e

$$\pi^e = \frac{k}{1 + \delta}$$

plug in $\pi = \frac{\pi^e - \varepsilon + k}{2 + \delta}$

$$\pi = \frac{\frac{k}{1 + \delta} - \varepsilon + k}{2 + \delta}$$

to get the equilibrium inflation rate

$$\pi = \frac{k}{1 + \delta} - \frac{\varepsilon}{2 + \delta}$$

Optimal degree of conservatism

Plug π^e and π in the utility function of the society $U = -(\pi - \pi^e + \varepsilon - k)^2 - \pi^2$ (after having used the definition of the supply curve)

$$U = -\left(\frac{k}{1 + \delta} - \frac{\varepsilon}{2 + \delta} - \frac{k}{1 + \delta} + \varepsilon - k\right)^2 - \left(\frac{k}{1 + \delta} - \frac{\varepsilon}{2 + \delta}\right)^2$$

rewrite

$$U = -\left(\frac{1 + \delta}{2 + \delta}\varepsilon - k\right)^2 - \left(\frac{k}{1 + \delta} - \frac{\varepsilon}{2 + \delta}\right)^2$$

take expectation, knowing that the element with ε are zero and the elements with ε^2 become σ_ε^2

$$\mathbb{E}U = - \left(\left(\frac{1+\delta}{2+\delta} \right)^2 \sigma_\varepsilon^2 + k^2 \right) - \left(\left(\frac{k}{1+\delta} \right)^2 + \frac{\sigma_\varepsilon^2}{(2+\delta)^2} \right)$$

group similar elements (i.e. σ_ε^2 and k^2)

$$\mathbb{E}U = - \left[[(1+\delta)^2 + 1] \left(\frac{\sigma_\varepsilon^2}{(2+\delta)^2} + \frac{k^2}{(1+\delta)^2} \right) \right]$$

maximize the utility, take first derivative wrt δ , and set it equal to zero

$$- \left[\begin{array}{l} 2(1+\delta) \left(\frac{\sigma_\varepsilon^2}{(2+\delta)^2} + \frac{k^2}{(1+\delta)^2} \right) \\ + [(1+\delta)^2 + 1] \left(-\frac{2\sigma_\varepsilon^2}{(2+\delta)^3} - \frac{2k^2}{(1+\delta)^3} \right) \end{array} \right] = 0$$

rewrite

$$-(1+\delta) \left(\frac{\sigma_\varepsilon^2}{(2+\delta)^2} + \frac{k^2}{(1+\delta)^2} \right) + [(1+\delta)^2 + 1] \left(\frac{\sigma_\varepsilon^2}{(2+\delta)^3} + \frac{k^2}{(1+\delta)^3} \right) = 0$$

rewrite

$$\frac{[(1+\delta)^2 + 1] \sigma_\varepsilon^2}{(2+\delta)^3} - \frac{(1+\delta) \sigma_\varepsilon^2}{(2+\delta)^2} + \frac{[(1+\delta)^2 + 1] k^2}{(1+\delta)^3} - \frac{(1+\delta) k^2}{(1+\delta)^2} = 0$$

which gives

$$\delta \frac{\sigma_\varepsilon^2}{(2+\delta)^3} = \frac{k^2}{(1+\delta)^3}$$

or simplified

$$\delta = \frac{k^2 (2+\delta)^3}{\sigma_\varepsilon^2 (1+\delta)^3}$$

Road map (Sidrauski → NK)

□ Chapter 2

I) intertemporal utility maximization s.t. budget constraint



first order conditions (FOC)



main features (essentially at the steady state):

Neutrality, Superneutrality, Money demand, Cost of inflation, Interest on money

□ Chapter 2

II) intertemporal utility maximization s.t. budget constraint (Sidrauski, 7.2 script)



first order conditions (FOC)



representation of the model w/o functional forms (TABLE 13)



representation of the model with functional forms (TABLES 14/15)



linearization (TABLE 16/17)

□ Chapter 5

III) simplification of the model (TABLES 18/19/20)



monopolistic competition and price rigidity (Calvo)

□ Chapter 5

IV) intertemporal utility maximization s.t. budget constraint (NK, 8.4 script)



first order conditions (FOC)



NK (TABLES 22/23)

Nationale Buchhaltung

- Produktionsansatz Y , Summe der Wertschöpfung
- Verwendungsansatz, $C + I + G + NX$
- Einkommenansatz, Summe der Einkommenströme

Cobb-Douglas

- Production function of the single good y :

$$y_t = k_{t-1}^\alpha (e^{z_t} l_t)^{1-\alpha} \quad \text{with} \quad z_t = \rho z_{t-1} + \varepsilon_t \quad (1)$$

at the steady state $y = k^\alpha l^{1-\alpha}$

- i) Substitution: both factors needed for production (\rightarrow non separable)
- ii) Economies of scale: $\alpha + 1 - \alpha = 1$ (constant returns to scale)

$$\begin{aligned} & (xk_{t-1})^\alpha (e^{z_t} xl_t)^{1-\alpha} \\ & x^\alpha k_{t-1}^\alpha x^{1-\alpha} (e^{z_t} l_t)^{1-\alpha} \\ & xk_{t-1}^\alpha (e^{z_t} l_t)^{1-\alpha} \\ & xy \end{aligned}$$

- iii) Labor share: perfect competition implying that factor price = marginal product of factor, e.g. $w_t = MPL_t$

$$\frac{w_t l_t}{y_t} = \frac{MPL_t l_t}{y_t} = \frac{((1 - \alpha) k_{t-1}^\alpha (e^{z_t})^{1-\alpha} l_t^{-\alpha}) l_t}{k_{t-1}^\alpha (e^{z_t} l_t)^{1-\alpha}} = 1 - \alpha$$

- iv) Technical rate of substitution: assuming $dz_t = 0$ and the total differential $dy_t = 0$

$$dy_t = \alpha k_{t-1}^{\alpha-1} (e^{z_t} l_t)^{1-\alpha} dk_{t-1} + (1 - \alpha) k_{t-1}^{\alpha} (e^{z_t})^{1-\alpha} l_t^{-\alpha} dl_t = 0$$

TRS_t

$$\frac{dk_{t-1}}{dl_t} = -\frac{1 - \alpha}{\alpha} \frac{k_{t-1}}{l_t}$$

- v) Elasticity of substitution: using $|TRS_t| = \frac{1-\alpha}{\alpha} \frac{k_{t-1}}{l_t}$; rewrite TRS_t -equation using the logarithm: $\ln \frac{k_{t-1}}{l_t} = \ln |TRS_t| - \ln \frac{1-\alpha}{\alpha}$; use the formula for the elasticity of substitution

$$\frac{d \ln \frac{k_{t-1}}{l_t}}{d \ln |TRS_t|} = 1$$

Interest rate structure: example

3 markets

Y1: 4%

Y2: 8%

2Y: ?

$$\begin{aligned} \left(1 + \frac{4}{100}\right) \left(1 + \frac{8}{100}\right) &= \left(1 + \frac{12.32}{100}\right) \\ \left(1 + \frac{?}{100}\right)^2 &= \left(1 + \frac{12.32}{100}\right) \\ ? &= \left(\sqrt{1 + \frac{12.32}{100}} - 1\right) 100 = 5.9811 \end{aligned}$$

Strategy if ? < 6% (no arbitrage condition): 2Y-loan (demand ↗, interest ↗), Investition Y1 und Y2 (Angebot ↗, interest ↘)

Solution Example Question 9

□ Price-elasticity of demand $Y_t(i)$

$$Y_t(i) = \left(\frac{P_t}{P_t(i)} \right)^{\frac{1}{1-q}} Y_t$$

– solution, quick way

$$\begin{aligned} \ln Y_t(i) &= \frac{1}{1-q} \ln P_t - \frac{1}{1-q} \ln P_t(i) + \ln Y_t \\ \frac{\partial \ln Y_t(i)}{\partial \ln P_t(i)} &= -\frac{1}{1-q} \end{aligned}$$

– solution, long way

$$\begin{aligned} \frac{\partial Y_t(i)}{\partial P_t(i)} \frac{P_t(i)}{Y_t(i)} &= \overbrace{\left[\frac{1}{1-q} \left(\frac{P_t}{P_t(i)} \right)^{\frac{1}{1-q}-1} Y_t \frac{-P_t}{P_t(i) P_t(i)} \right]}^{-\frac{1}{1-q} \frac{Y_t(i)}{P_t(i)}} \frac{P_t(i)}{Y_t(i)} \\ &= \left[\frac{1}{1-q} \left(\frac{P_t}{P_t(i)} \right)^{\frac{1}{1-q}-1} Y_t \frac{-P_t}{P_t(i) P_t(i)} \right] \frac{P_t(i)}{\left(\frac{P_t}{P_t(i)} \right)^{\frac{1}{1-q}} Y_t} \\ &= -\frac{1}{1-q} \left(\frac{P_t}{P_t(i)} \right)^{\frac{1}{1-q}-1} \left(\frac{P_t}{P_t(i)} \right)^{-\frac{1}{1-q}} \frac{Y_t P_t(i)}{Y_t P_t(i)} \frac{P_t}{P_t(i)} \\ &= -\frac{1}{1-q} \end{aligned}$$

□ Elasticity of substitution of production function Y_t

$$Y_t = \left[\int Y_t(i)^q \mathbf{d}i \right]^{\frac{1}{q}}, 0 < q \leq 1$$

– rewrite

$$Y_t = [\dots + Y_t(i_1)^q + Y_t(i_2)^q + \dots]^{\frac{1}{q}}$$

– total differential

$$\begin{aligned} dY_t &= [\dots + Y_t(i_1)^q + Y_t(i_2)^q + \dots]^{\frac{1}{q}-1} \\ &\times [\dots + Y_t(i_1)^{q-1} dY_t(i_1) + Y_t(i_2)^{q-1} dY_t(i_2) + \dots] \end{aligned}$$

– restrictions

$$dY_t = dY_t(i_3) = dY_t(i_4) = \dots = 0$$

– apply to dY_t to get the TRS

$$\begin{aligned} 0 &= Y_t(i_1)^{q-1} dY_t(i_1) + Y_t(i_2)^{q-1} dY_t(i_2) \\ \frac{dY_t(i_1)}{dY_t(i_2)} &= - \left(\frac{Y_t(i_2)}{Y_t(i_1)} \right)^{q-1} \end{aligned}$$

– compute $\ln |TRS|$

$$\begin{aligned} (q-1) \ln \frac{Y_t(i_2)}{Y_t(i_1)} \\ (1-q) \ln \frac{Y_t(i_1)}{Y_t(i_2)} \end{aligned}$$

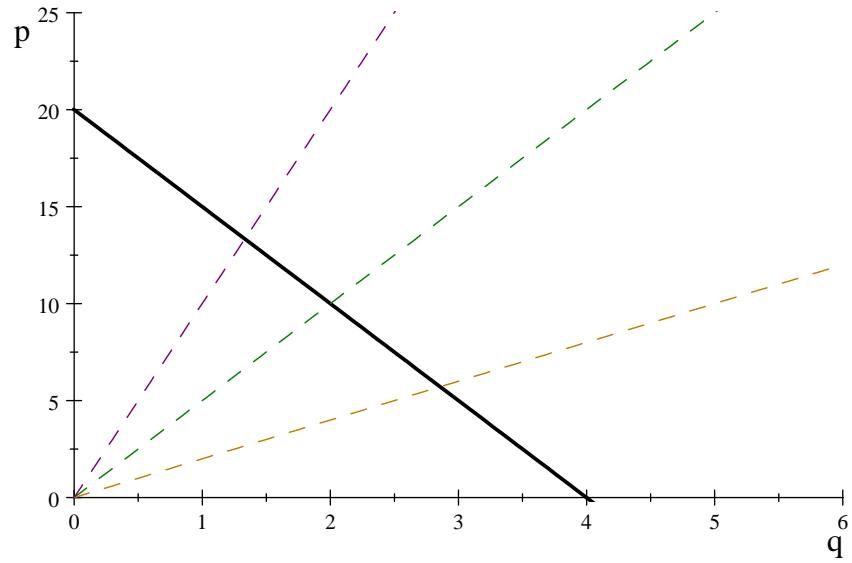
– compute $\frac{d \ln |TRS_t|}{d \ln \frac{Y_t(i_1)}{Y_t(i_2)}}$

$$1 - q$$

– elasticity of substitution

$$\frac{d \ln \frac{Y_t(i_1)}{Y_t(i_2)}}{d \ln |TRS_t|} = \frac{1}{1 - q}$$

□ Elasticity, graphical representation, $\frac{\partial q}{\partial p} \frac{p}{q} = \frac{p}{\frac{q}{\partial p}}$



Solution Example Question 11

- Random walk, AR(1), $x_t = \alpha + \beta x_{t-1} + \varepsilon_t$, with unit root, $\beta = 1$

$$x_t = \alpha + x_{t-1} + \varepsilon_t$$

$$\mathbb{E}x_{t+1} = \alpha + x_t + \mathbb{E}\varepsilon_{t+1} = \alpha + x_t$$

- Growth rates

		qoq ann	yoy
Feb-05	47.73		
May-05	51.65	32.85	
Aug-05	61.62	77.21	
Nov-05	56.87	-30.83	
Feb-06	61.82	34.82	29.52
May-06	69.60	50.34	34.75
Aug-06	69.67	0.40	13.06
Nov-06	59.65	-57.53	4.89
Feb-07	57.79	-12.47	-6.52
May-07	68.69	75.45	-1.31
Aug-07	74.74	35.23	7.28
Nov-07	88.63	74.34	48.58
Feb-08	96.67	36.29	67.28
May-08	116.00	79.98	68.87
Aug-08	116.00	0.00	55.20
Nov-08	116.00	0.00	30.88
Feb-09	116.00	0.00	20.00
May-09	116.00	0.00	0.00
Aug-09	116.00	0.00	0.00
Nov-09	116.00	0.00	0.00
Feb-10	116.00	0.00	0.00
May-10	116.00	0.00	0.00
Aug-10	116.00	0.00	0.00
Nov-10	116.00	0.00	0.00
Feb-11	116.00	0.00	0.00
May-11	116.00	0.00	0.00
Aug-11	116.00	0.00	0.00
Nov-11	116.00	0.00	0.00