Economics of central banking (Zentralbanktheorie und Zentralbankpolitik)

Supplementary slides: Envelope theorem

Nicolas A. Cuche-Curti, Swiss National Bank and University of St. Gallen

nicolas.cuche-curti@snb.ch http://cuche.net/classes.htm

These slides present the main elements of the course 7,280; they are based on several papers and books (e.g. Walsh, 2003, Galí, 2008) whose references are given during the term; a separate reading list for the exam will be given after the term break.

November 29, 2009

Envelope theorem

- Theorem

$$M(a) = \max_x f(x,a)$$
$$\frac{dM(a)}{da} = \left.\frac{\partial f(x,a)}{\partial a}\right|_{x^*=x^*(a)}$$

- Example, setup
  - main variables in a profit maximization problem
    - p=output price
    - x=output
    - w=wage/hour
    - L=hours worked
    - $\Re$=profit
  - firms uses the following profit function, turnover minus cost, which they maximize finding the optimal output, i.e. the optimal labor demand
    $$\Re(p,x,w,L) = px - wL$$
  - technology is given by the production function
    $$x = \sqrt{L}$$
  - firms behave as a price-taker on the labor market
    $$w = 1$$

- profit function with production function and wage assumption

$$\Re(p, L) = p\sqrt{L} - L$$

- maximize profit $\Re$ with respect to L, first derivative equals zero; it gives an optimal labor demand as function of the output price p

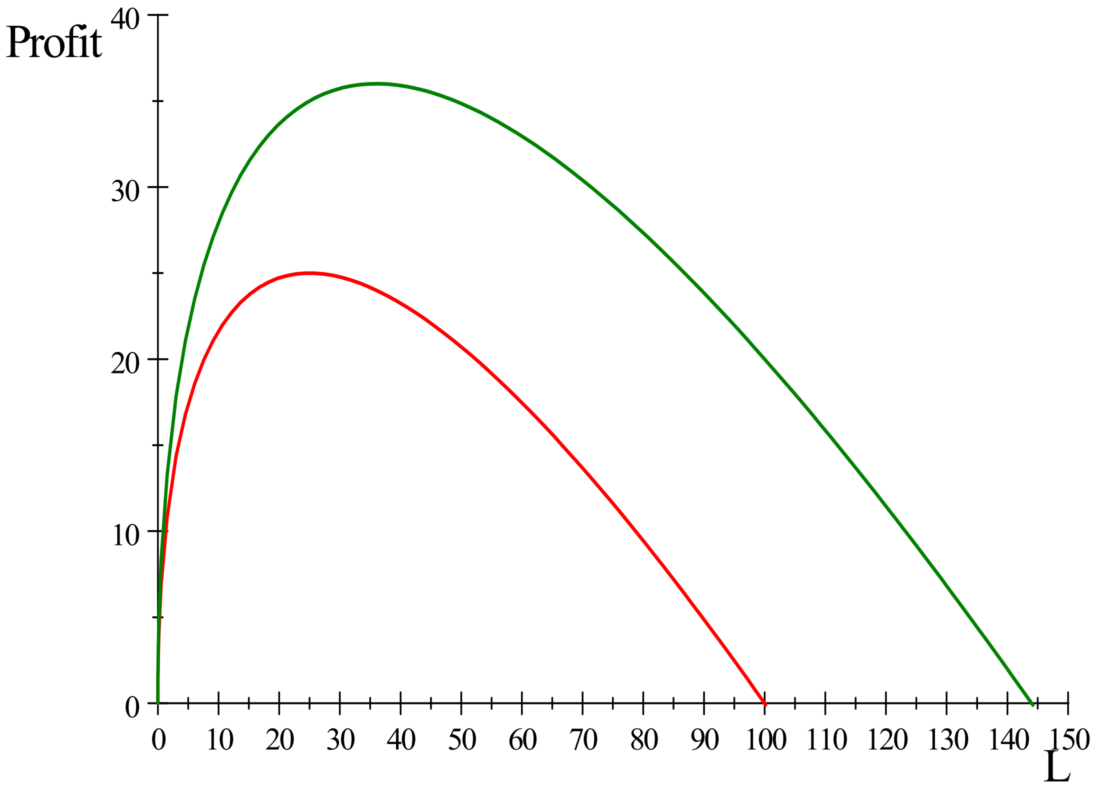
$$0 = p\frac{1}{2}L^{-\frac{1}{2}} - 1 \qquad \rightarrow \qquad L^\star(p) = \frac{p^2}{4}$$

- level of maximized profit ($\Re^\star$ is called the value function)

$$\Re^\star(p) = \max_L \Re(p, L) = \Re(L^\star(p), p)$$

$$\Re^\star = p\sqrt{L^\star} - L^\star = p\sqrt{\frac{p^2}{4}} - \frac{p^2}{4} = \frac{p^2}{4}$$

- plot of two profit functions, one with a price $p = 10$ (red line) and one with a price $p = 12$ (green line); note that the fact that $L^\star$ and $\Re^\star$ are the same is random



- Example, question
  - how does the optimal profit $\Re^\star$ change, when the price p changes?
    - ▷ if possible, direct calculation

$$\frac{d\Re^\star}{dp} = \frac{d\left(\frac{p^2}{4}\right)}{dp} = \frac{p}{2}$$

▷ otherwise, use the envelope theorem

$$\frac{d\Re^\star}{dp} = \left.\frac{\partial \Re}{\partial p}\right|_{L^\star} = \left.\sqrt{L}\right|_{L^\star} = \sqrt{\frac{p^2}{4}} = \frac{p}{2}$$

□ Example, some intuition

– maximized profit, value function

$$\Re^\star(p) = \max_L \Re(p, L) = \Re(L^\star(p), p)$$

$$\Re^\star = p\sqrt{L^\star} - L^\star$$

– differential of the value function

$$\frac{d\Re^\star}{dp} = \frac{\partial \Re(L^\star(p), p)}{\partial L}\frac{\partial L^\star(p)}{\partial p} + \frac{\partial \Re(L^\star(p), p)}{\partial p}$$

$$\frac{d\Re^\star}{dp} = \frac{\partial \Re(L^\star(p), p)}{\partial p} \qquad \rightarrow \qquad \frac{d\Re^\star}{dp} = \left.\frac{\partial \Re}{\partial p}\right|_{L^\star}$$

– applied to the example

$$\frac{d\Re^\star}{dp} = \left(p\frac{1}{2}\left(L^\star\right)^{-\frac{1}{2}} - 1\right)\frac{\partial L^\star(p)}{\partial p} + \sqrt{L^\star} = 0 + \sqrt{L^\star} = \frac{p}{2}$$

□ Application to utility maximization

– value function where $\omega_t = c_t + k_t + b_t + m_t$

$$V(\omega_t) = \max_{c_t, k_t, b_t, m_t} \left\{u(c_t, m_t) + \beta V(\omega_{t+1})\right\} \qquad \text{(2.5 Walsh)}$$

– shorter version

$$V(\omega) = \max_c u$$

– derivates, part of the FOC

$$V'_\omega = u'_\omega\Big|_{c^\star = c^\star(\omega)}$$

$$V'_\omega = u'_c(c^\star)\frac{dc}{d\omega}$$

$$V'_\omega = u'_c(c^\star)$$

and because we are in optimizing framework, it gives

$$V'_\omega = u'_c$$